Method

The finite difference solution of Laplace's Equation gives an approximation to the exact potential function at a finite number of mesh or nodal points within the given boundaries of the transmission line conductors. If a suitable interpolation formula is used to define an approximate, but continuous potential distribution throughout the region, the associated field energy may be calculated. By the Dirichlet principle, or the principle of minimum potential energy, it is known that this energy is greater than the energy E associated with the exact potential function V. It follows that the calculated capacitance per unit length, which is proportional to the field energy, is greater than the exact capacitance per unit length C.

The characteristic impedance Z_0 of a TEM mode transmission line is given by

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{vC}$$

where L is the inductance per unit length and v is the velocity of propagation. A lower bound on the characteristic impedance has thus been obtained.

The dual problem is defined by interchanging electric conductors (short circuit) and magnetic conductors (open circuits) and substituting the reciprocal of the dielectric constant K_e.

$$C' = \frac{2E'}{V_0'^2}.$$

An exact dual potential function V' can be shown to be related to function V of the original problem by the transformation

$$K_e \frac{\partial V}{\partial x} = \frac{\partial V'}{\partial y}, \qquad K_e \frac{\partial V}{\partial y} = -\frac{\partial V'}{\partial x}. \quad (1)$$

Hence

$$E = \frac{\epsilon_0 K_e}{2} \int\int_R \left\{ \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 \right\} dxdy$$
$$= \frac{\epsilon_0}{2K_e} \int\int_R \left\{ \left(\frac{\partial V'}{\partial y}\right)^2 + \left(\frac{\partial V'}{\partial x}\right)^2 \right\} dxdy$$
$$= E'$$

where E' is the energy in the dual problem and R is the region between the transmission line conductors. Also

$$V_0' = \int_l \frac{\partial V'}{\partial s} ds = \int_l K_e \frac{\partial V}{\partial n} ds = -\frac{Q}{\epsilon_0}$$

where Q is the charge on unit length of the transmission line conductor enclosed by curve l. Thus the exact dual capacitance per unit length is given by

$$C' = \frac{2E'}{V_0'^2} = \frac{2E\epsilon_0^2}{Q^2} = \frac{\epsilon_0^2}{C}.$$

Hence

$$\frac{C'}{\epsilon_0} = \frac{\epsilon_0}{C}, \quad (2)$$

where ϵ_0M8.8542×10^{-12} farad per meter is the permittivity of free space.

If an approximate, but continuous dual potential distribution is found, the associated field energy is greater than the energy E' associated with the exact potential function V'. It follows that the calculated dual capacitance is greater than the exact dual capacitance C'. Hence by (2), a lower bound on the exact capacitance C and thus an upper bound on the characteristic impedance Z_0 have been obtained.

This procedure is analogous to calculating the inductance L of the transmission line with relative permeability K_m equal to $1/K_e$, but its real significance lies in the construction of an approximate, but solenoidal electric field vector, which is to be discussed in a forthcoming paper [4].

The approximate, but continuous dual potential distribution may be obtained directly from the finite difference solution of the original problem with the aid of the transformation (1). This latter method for an upper bound eliminates the need to set up and solve the dual problem, and it was incorporated in the Laplace finite difference computer program described by H. E. Green [3] together with the above method for a lower bound. The following examples illustrate typical results which have been obtained.

Results

a) Square Coaxial Line (Side Length Ratio = 2).

b) Strip Transmission Line ($w/b = 0.8$; $g/b = 3.2$), see Fig. 1.

Problem Number	Number of Nodes in the Finite Difference Net	Characteristic Impedance (Ohms)			
		Lower Bound	Upper Bound	Mean of Upper and Lower	Exact Impedance*
a	441	36.6942	36.9524	36.8229	36.81132
a	1682	36.7636	36.8656	36.8146	
a	6561	36.7921	36.8316	36.8119	
a	Extrapolated to infinity	36.8026	36.8192	36.8109	
b	363	74.4665	77.8513	76.1213	75.9079
b	1365	75.1702	76.8896	76.0202	
b	5289	75.5271	76.3939	75.9580	
b	Extrapolated to infinity	75.6628	76.2072	75.9340	

* The exact impedance was obtained from a conformal transformation.

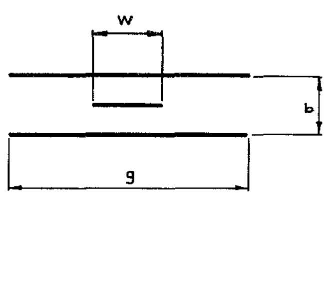

Fig. 1. Strip transmission line.

Conclusion

A method of extending a Laplace finite difference solution to obtain an upper and a lower bound on the characteristic impedance of TEM mode transmission lines has been demonstrated.

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Correction to "General Four-Resonator Filters at Microwave Frequencies"

R. M. Kurzrok, author of the above,[1] has called the following to the attention of the Editor.

On page 296, the first sentence below (4) should have read:

"Letting $|x_v| = 2.75$ and using (4), a theoretical valley insertion loss of 34.4 dB is obtained."

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[1] R. M. Kurzrok, *IEEE Trans. on Microwave Theory and Techniques (Correspondence)*, vol. MTT-14, pp. 295–296, June 1966.

Addendum to Analysis and Exact Synthesis of Cascaded Commensurate Transmission-Line C-Section All-Pass Networks

In a previous publication [1] a method for the exact synthesis of cascaded commensurate transmission-line C-section all-pass networks was presented. In the general case of n sections, the synthesis procedure requires the solution of a set of n simultaneous linear equations before extracting the even-mode impedances of the coupled lines

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of the cascaded C-sections. Further study has since shown that it is unnecessary to solve, or even ascertain, the aforementioned set of simultaneous equations; that the essential information for the synthesis procedure is contained in the coefficients of the phase function, itself. The phase function β for cascaded C-section all-pass networks is given by [1]

$$\beta = 2 \angle F_n(s) = 2 \angle [N_n(s) + D_n(-s)], \quad (1)$$

where the symbol $\angle$ stands for "the angle of"; the subscript n is the number of cascaded sections; and $N_n(s)$ and $D_n(s)$ are the numerator and denominator, respectively, of the reflection coefficient of the corresponding transformer prototype, terminated in a 1-ohm resistor. It can be shown that

$$N_n(s) + D_n(-s) = F_n(s) = 2(A - C), \quad (2)$$

where A and C are the even and odd polynomials, respectively, that constitute the A and C components of the overall ABCD-matrix for the transformer prototype. Hence,

$$A \doteq \text{Even part of } F_n(s) = F_e(s) \quad (3)$$

$$C \doteq -\text{ Odd part of } F_n(s) = -F_0(s). \quad (4)$$

In the case where the transformer prototype is terminated not in a 1-ohm load but in an open circuit, the input impedance is given by

$$Z_{\text{in}}(s)\,|_{R_L=\infty} = A(s)/C(s) = -F_e(s)/F_0(s). \quad (5)$$

Thus, the even-mode impedances of the cascaded C-sections may be extracted from (5) directly, without resorting to the solution of the aforementioned simultaneous equations. The realization of the reactance function of (5) in a cascade of commensurate transmission lines terminated in an open circuit is guaranteed by a theorem of Richards [2].

To illustrate the use of (5), we use the phase function $F_3(s)$ which was previously presented [1] in the design of a 3-section 90° phase shifter.

$$F_3(s) = 1 - 1.8s + 1.57256s^2 - 0.41763s^3. \quad (6)$$

By (5), one finds directly

$$Z_{\text{in}}(s)\,|_{R_L=\infty} = \frac{1 + 1.57256s^2}{1.8s + 0.41763s^3}. \quad (7)$$

The line impedances may be extracted by the procedure of Richards [2] yielding the same values as previously found [1].

Also in this correspondence, we wish to emphasize that the restrictions on the coefficients of the phase function (see (38) and (39) of [1]) are necessary but not sufficient (except for the case $n=1$) to realize cascaded C-sections with even-mode impedances greater than 1. For two sections necessary and sufficient conditions have been found to be

$$\left.\begin{array}{ll} & 0 < B_1 \le 2 \\ \text{For} & 0 < B_1 \le 1, \quad B_2^2 \text{ unrestricted} \\ \text{For} & 1 \le B_1 \le 2 \\ & B_1 - 1 \le B_2^2 \le \dfrac{1}{B_1 - 1}. \end{array}\right\} \quad (8)$$

For more than two sections sufficient conditions have not been determined. Nevertheless, in most cases of practical interest, it is believed that the synthesis method will yield even-mode impedances greater than one. In this respect the situation is analogous to that encountered in the synthesis of directional couplers from prescribed insertion loss functions. There, also, it is not known beforehand which insertion loss functions will realize even-mode impedances greater than one.

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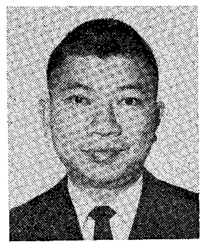

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